On driving a viscous fluid out of a tube

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Two problems are considered. First, it is shown experimentally that the amount of viscous fluid left on the walls of a horizontal tube, when it is expelled by an inviscid fluid, reaches an asymptotic value of 0·60 of the amount required to fill the tube, when the parameter $\mu U/T$ is increased, μ and T being the coefficients of viscosity and interfacial surface tension respectively, and U the velocity of the interface between the two fluids. Secondly, by neglecting the inertia terms in the equations of motion and the effect of gravity, a theory for the passage of this type of bubble is presented, together with experimental results in support of the theory. It is shown that such a solution is only valid under certain other restrictions, and then only to within half a tube diameter of the nose of the bubble.

1. Introduction

For some years there has been a certain amount of interest in the problems of bubbles of air or liquid in fluid contained in a tube, and this present work is an extension of that undertaken in this field by Taylor (1961). In that paper he discussed the fractional amount of viscous fluid which is left on the wall of a tube when it is expelled by a column of inviscid fluid. This fraction m depends on the non-dimensional parameter $\mu U/T$ where μ is the coefficient of viscosity, U the velocity of the interface between the two fluids, and T the coefficient of interfacial surface tension. His results led him to suggest that m would tend to an asymptotic value as $\mu U/T$ increased, and it was thought to be of interest to investigate this point experimentally. It was found (§ 2), by using Golden Syrup, and a photographic technique, that there was indeed an asymptotic value which was in good agreement with that suggested by Taylor. This type of approach required an axisymmetric bubble,† and under these conditions the radius of the bubble also tended to an asymptotic value for large values of $\mu U/T$, since the ratio λ of bubble width to tube width is related to the fractional amount left in the tube by the simple expression $m = 1-\lambda^2$.

In § 3, an attempt is made to describe analytically the motion of an axisymmetric bubble, by using the equations for slow motion of a viscous fluid, and by neglecting the effects of gravity, and the results of this are compared with experiment in the following sections.

† In what follows the word 'bubble' is used to describe the interface between the two fluids, although this term is not quite appropriate to an interface which is not a closed surface.

2. Initial experiments

2.1. *General method*

A diagram of the apparatus is shown in figure 1, and is almost self-explanatory. It was found necessary to have two jackets round the 'Veridia' (accurate bore) glass tubing T in which the bubble was formed, the first J to compensate for refractive effects as far as possible by being filled with Golden Syrup, and the second being part of the water jacket W which completely surrounded the apparatus, thus ensuring a constant temperature. This latter jacket was essential due to the very marked dependence of the viscosity of the syrup on temperature.

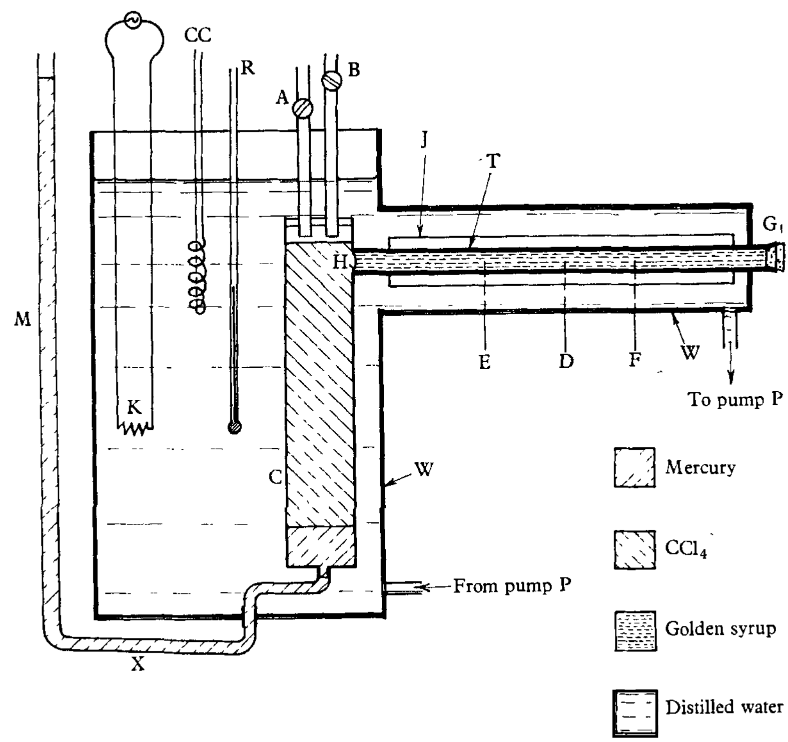

FIGURE 1. Schematic diagram of the apparatus, immediately prior to the passage of a bubble.

It was filled with distilled water for optical reasons. The temperature of the bath was kept constant (to within 0·1 °C) by using a contact thermometer R coupled to an electronic relay and controlling a heater K whose heat output could be quickly and easily controlled by a rheostat. This allowed the operating temperature to be quickly attained, and the fluctuations from this temperature to be reduced to a minimum—probably well within the limits stated above. Thorough circulation and stirring was guaranteed by the small centrifugal pump P. A cooling coil of copper tubing, CC, carrying tap water, made it possible to adjust the working temperature to any predetermined value.

Photographs were taken in the horizontal plane (see figure 2) at the point marked D with flashgun and plate camera, and the velocity was measured by stop watch at E and F, thus checking whether the bubble was departing from uniform translation. Due to the colour of the syrup it was necessary to use panchromatic plates, and Ilford H.P. 3 gave excellent results.

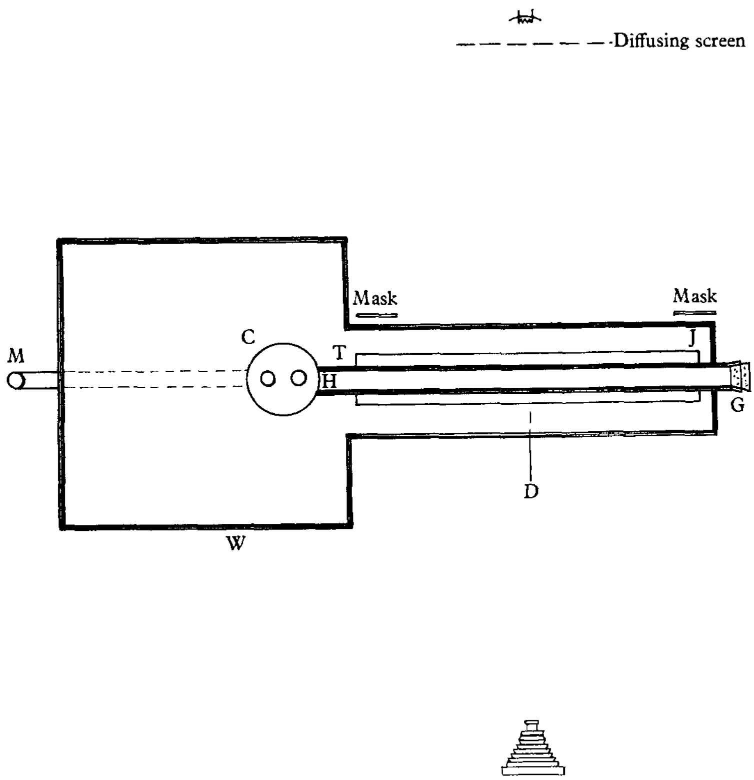

FIGURE 2. Plan of apparatus showing photographic arrangements.

The procedure was as follows. The pressure-head column M was filled with mercury, and adjusted so that the base of the Perspex cylinder C was covered. Carbon tetrachloride was poured in through tube A to a level just below that of the glass tube, T. Next, taking care to avoid air bubbles, Golden Syrup was added by way of tube B. This syrup, being less dense than CCl_4, lay on the surface of the latter and gradually made its way down the horizontal tube, *T*, until the tube was completely full. A rubber bung was placed in the end G. The pump, heater, etc., were then switched on and after the operating temperature set on the contact thermometer had been reached, the apparatus was left for about three hours to attain equilibrium. Then with tap B closed and tap A open, the pressure was raised in the mercury column until the CCl_4—Golden Syrup interface was above the level of the tube entrance H, excess fluid and some air being driven out through A. Tap A was now closed, and the pressure increased still further to an empirically calculated value. The experiment was started by removing the cork at G, the photograph taken and velocities measured as the bubble passed down the tube. An example of a bubble photographed by the above method is reproduced in figure 3. Subsequent experiments were more quickly made, for, once the tube had been removed, very thoroughly cleaned, and replaced, the CCl_4 in the cylinder could be topped up, and the tube refilled as before. From then on the procedure was exactly as above.

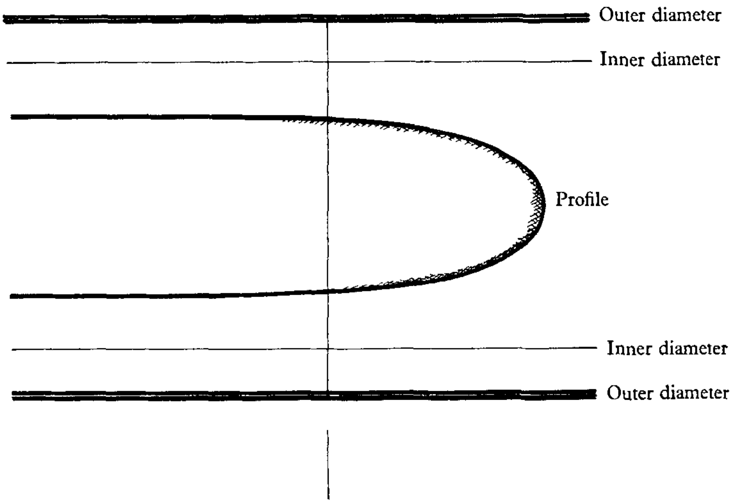

FIGURE 3. Typical bubble profile, moving with velocity 0·50 cm/sec. Parameter $\mu U/T = 12{\cdot}5$. Actual tube bore 1·00 cm. The vertical line appearing in the centre of the photograph is one of a series of marks scratched on the jacket J to measure the velocity of the bubble.

The amount of fluid left behind in the tube could then be found from measurements taken on the photographic negative. This was carried out by measuring the inner and outer diameters of the tube and the width of the bubble, at regularly increasing distances from the nose of the bubble. This work was facilitated by using a travelling microscope with a two-way travel, and capable of $\pm$ 0·001 mm accuracy. In all photographs it was apparent that the bubble width tended to an asymptotic value within about $1\frac{1}{2}$ tube diameters of the nose (see figure 3). This fact was quite discernible from the measurements taken as above. The asymptotic width of the bubble was calculated by taking the mean of about twenty measurements from the region where this was permissible. The mean value of the outer diameter of the tube was also calculated, and about 40 values could be used for this as it did not depend on the position of the bubble. As the actual ratio of the two diameters had been measured on the tube used in the experiment, the true value of the inner diameter, in the scale of the photograph, could be calculated by simple proportion from the mean outer diameter. This lengthy procedure was necessary because of the distortion of the inner diameter in a photograph. For some further detail on this point see § 2.2.2.

The ratio λ of asymptotic bubble width to inner diameter, and thus m, the fractional amount of viscous fluid left in the tube, were then found. The accuracy of the measurements made on the microscope was limited by the definition obtainable in a photograph. Hence, it was desirable to take the mean of as many measurements as possible to find the bubble width and outer tube diameter, before calculating the value of λ.

The parameter $\mu U/T$ was found by measuring the three quantities involved independently; U as above, by taking the mean of the two measured velocities; μ and T as below. Several experiments were made at different values of $\mu U/T$, all greater than 10, but the photographs obtained were completely indistinguishable, implying exactly the same value of λ, and hence of m, this latter being 0·60.

2.2. *Notes on the experiments*

2.2.1. *Choice of fluids.* It was at first thought that some sort of oil would be suitable for the viscous fluid, but the only oils with high enough viscosity at room temperature were far too opaque for photography. It was recently realized that silicone oils might have proved very satisfactory, but the problem of thorough cleaning of the tube would have remained. Hence Golden Syrup was chosen, and providing a reasonably fresh sample was used, the results were consistent. As a sample aged, non-Newtonian properties became evident, but as the colour of syrup darkens with age, a rough check could be kept on this point.

The choice of carbon tetrachloride as the driving fluid was influenced by several factors. These were first, that it is inviscid compared with Golden Syrup (the ratio of the viscosities at 20 °C being 1 to 40,000 approximately); secondly, that it is immiscible with mercury and with the syrup; and third, that although not a perfect match as regards density, it is very nearly so, thus avoiding possible asymmetry due to gravity forces. There was, in fact, no detectable asymmetry of the profile even when the photographic negative was examined under the travelling microscope, and the measurements described above verified this. That CCl_4 is slightly heavier than Golden Syrup greatly facilitated the filling of the apparatus.

It was later realized that CCl_4 attacks Perspex to a certain degree, and some surface 'crazing' became apparent after a month or so. However this disadvantage by no means outweighed the advantage of having certain parts of the equipment made from Perspex due to the excellent optical properties of this material. (Further difficulties with Perspex occurred due to residual stress in machined parts. This is an unavoidable possibility unless machined articles are subjected to special treatment.)

2.2.2. *Refractive index and distortion.* Since the refractive indices of CCl_4, Golden Syrup, and glass are slightly different, the photographs taken must be subject to a certain amount of distortion caused by the cylindrical-lens effect of the glass tube. The effects of distortion were kept to a minimum by enclosing the tube in a jacket containing Golden Syrup. This did not eliminate it entirely by any means, and the geometrical optics of the system were still quite complicated. Various light rays passing through the tube were considered, and the effects calculated. Two methods were used, first, a graphical ray-tracing technique, and secondly an analytic method, and the results obtained were in good agreement. In general, it was found that there was a negligible amount of distortion of the bubble profile, but an appreciable amount (1·7 %) at the inner edge of the glass tube. But the effect of distortion as regards calculating results, even at this inner edge of the tube, could be avoided by finding the true inner diameter of the tube from the outer diameter by simple proportion, as described above. In the course of measuring the outer diameter and the bubble width on the photograph, a note of the inner diameter was made, so that this could be compared with the value obtained by proportion, and the distortion thereby found. This gave a check on the variation of the refractive indices since, by considering the optics of the system, there is a simple relation between the distortion of the inner diameter and the ratio of the indices of glass and syrup.

It was noticable in every photograph that the outer diameter of the tube and the bubble profile appeared as a double line (see figure 3). This phenomenon is easily explained by consideration of the refractive indices of the materials forming these boundaries, and it proved possible to find the ratio of the indices of the two substances forming the boundary by measuring the width of the double line, using a very elementary calculation. This was of value as it gave an immediate check on the optical properties of the media at the time of each photograph. This is particularly useful for substances like Golden Syrup whose properties are often in doubt, and which can vary very quickly with time.

2.2.3. *Viscosity.* As is to be expected in a fluid of high viscosity, there is a rapid decrease in the viscosity of Golden Syrup with increasing temperature, and hence care had to be taken in the measurement of it. The most satisfactory way of finding the value of the coefficient of viscosity at the operating temperature was to immerse a cylinder containing Golden Syrup (in fact the sample from which the tube was filled) into the water bath W and to measure the velocity of a ball-bearing falling down the axis of the cylinder. This gave consistent results, with an accuracy of 1 % for any particular sample of syrup, but did require a correction factor for the effect of the wall of the cylinder, as calculated by Haberman (1956).

2.2.4. *Surface tension.* Since reasonably accurate values of the interfacial surface tension of carbon tetrachloride and Golden Syrup were required, it was decided that the pendant drop method would be the easiest and most satisfactory method to apply. It did indeed give consistent results provided the drop was stationary. Experimentally the method was very similar to that used by Fordham (1948), again using the water bath to ensure measurement at the correct temperature, and the tables from the above paper were used to calculate the final values for the coefficient of interfacial surface tension. Consideration of the standard deviation from the mean of several measurements by this method gave a possible error of 2 % in the results at any particular temperature.

3. Theory

3.1. *Physical assumptions*

Two basic assumptions, justifiable on physical grounds, were made so that the analysis could be simplified. These were: (i) that the inertia terms in the equations of motion were negligible, and (ii) that gravity forces were also negligible. The first of these is justified by considering the Reynolds number based on the flow round the bubble (of asymptotic width λa). This parameter is $\rho_2 \lambda a U/\mu$, where ρ_2 is the density of the syrup, and under the experimental conditions described in this paper, this was of the order of 1/200. Obviously, for faster moving bubbles, or less viscous fluids, this assumption may not be valid, nor will it necessarily be valid where surface tension forces predominate. This latter case is discussed by Bretherton (1961).

Assumption (ii) implies that the bubble is axisymmetric. One way of considering the effect of gravity forces is to compare the time taken for the fluid left attached to the walls of the tube to drain to the lowest generator, with the time taken by the profile to advance a tube diameter. This is given by the parameter

$$(\rho_1 - \rho_2)\, g\lambda^2 a^2/\mu U,$$

where ρ_1 is the density of the inviscid fluid. Hence for the bubble to be axisymmetric, this parameter must be very much less than 1. From a practical point of view, the conditions of the experiments of § 2 give this parameter a value of 1/40, so it is reasonable that there should have been no detectable asymmetry. However

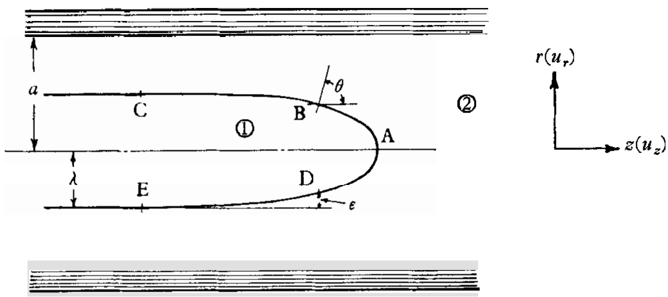

FIGURE 4. Section of bubble profile, and co-ordinate system.

for those experiments at 30 °C, as described below, where $\mu \simeq 90$ poise, the parameter is of the order of 0·2, so it is not surprising that slower bubbles at this temperature showed a detectable lack of symmetry. As far as the theory is concerned, this parameter is of little importance as it is theoretically possible to take fluids of exactly the same density. For this same reason the two conditions on the velocity of the interface, i.e. $U \ll \mu/\rho_2 \lambda a$ and $U \gg (\rho_1 - \rho_2)\, g\lambda^2 a^2/\mu$, do not lead to a contradiction.

3.2. *Exact equations*

Consider an inviscid fluid driving a viscous fluid along a tube (see figure 4). For the motion of a viscous fluid, when the inertia terms are negligible, the Navier–Stokes equation reduces to

$$\mu \nabla^2 \mathbf{u} - \operatorname{grad} p = 0, \tag{1}$$

and if the fluid is incompressible, this can be further reduced to

$$\operatorname{curl}(\nabla^2 \mathbf{u}) = 0. \tag{2}$$

Using cylindrical-polar co-ordinates (r, ϕ, z), take co-ordinate axes fixed in space, with the z-axis along the axis of the tube. Then by assumption (ii), all derivatives with respect to ϕ are zero. Hence the results given in Goldstein (1938, pp. 114–15) can be applied, i.e. there exists a stream function $\psi(r,z)$, such that the velocity u of the viscous fluid is given by

$$\mathbf{u} = (r^{-1}\,\partial\psi/\partial z, 0, \; -r^{-1}\,\partial\psi/\partial r),$$

and so, from (2), the equation for ψ is

$$\left(\frac{\partial^2}{\partial r^2}-\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}\right)^2 \psi(r,z) = 0. \tag{3}$$

The boundary conditions are as follows:

(i) the tangential and normal components of velocity of the fluid at the wall of the tube are zero;

(ii) the tangential stress on the interface between the fluids is zero;

(iii) the difference in the normal stress on the two sides of the interface is balanced by the surface-tension force;

(iv) there is no diffusion across the interface.

Expressing these conditions in analytic detail, condition (i), the no-slip condition, reduces immediately to

$$r^{-1}\,\partial\psi/\partial z = r^{-1}\,\partial\psi/\partial r = 0, \tag{4}$$

when $r = a$, if a is the radius of the tube.

By taking the expressions for the components of the stress tensor, and substituting for the velocities in terms of the above stream function, condition (ii) becomes

$$\frac{\sin 2\theta}{r}\left(2\frac{\partial^2\psi}{\partial r\,\partial z}-\frac{1}{r}\frac{\partial\psi}{\partial z}\right)-\frac{\cos 2\theta}{r}\left(\frac{\partial^2\psi}{\partial r^2}-\frac{1}{r}\frac{\partial\psi}{\partial r}-\frac{\partial^2\psi}{\partial z^2}\right) = 0, \tag{5}$$

on the interface, where θ is the angle between the outward normal to the surface of the bubble and the axis of z, as shown in figure 4.

Similarly, we obtain for condition (iii) that

$$p_1 - T\kappa = p_2+\mu\left\{\frac{\sin 2\theta}{r}\left(\frac{\partial^2\psi}{\partial r^2}-\frac{1}{r}\frac{\partial\psi}{\partial r}-\frac{\partial^2\psi}{\partial z^2}\right)+\frac{\cos 2\theta}{r}\left(2\frac{\partial^2\psi}{\partial r\,\partial z}-\frac{1}{r}\frac{\partial\psi}{\partial z}\right)+\frac{1}{r^2}\frac{\partial\psi}{\partial z}\right\}, \tag{6}$$

where p_1 is the (uniform) pressure in the inviscid fluid, p_2 is the pressure in the viscous fluid, T is the coefficient of interfacial surface tension, and κ is the mean curvature of the profile.

The final boundary condition, which is equivalent to the condition that the profile should be a streamline, is, in the notation of figure 4,

$$u_z \cos\theta + u_r \sin\theta = U\cos\theta,$$

where U is the velocity of the interface in the z-direction. This gives on substitution for u_z and u_r in terms of $\psi(r,z)$

$$-\frac{1}{r}\frac{\partial\psi}{\partial r}\cos\theta+\frac{1}{r}\frac{\partial\psi}{\partial z}\sin\theta = U\cos\theta \tag{7}$$

on the interface.

It should be noted that these conditions are those appropriate to axes fixed in space. Many authors, e.g. Haberman (1956), have approached similar problems by taking axes fixed with respect to the bubble. This involves small changes in conditions (i) and (iv) only.

3.3. *Approximations*

These equations, (3) to (7), although valid throughout the viscous fluid, and for any part of the profile, are not very easy to handle in their present general form, partly because the actual shape of the profile must be found as part of the solution.

Obviously the first simplifications to apply are those suggested by observation of a practical case. Let λ' represent the asymptotic radius of the bubble, which is known to exist by § 2, and let the axisymmetric profile be given by

$$r = \lambda' - \epsilon,$$

for some small positive perturbation ϵ. Further, as suggested by the asymptotic nature of the profile, and also by the results of fitting various curves to a typical interface, let ϵ be given by

$$\epsilon = \epsilon_0 \, e^{kz}, \quad \text{for some } \epsilon_0. \tag{8}$$

The validity of this equation is considered in § 4. Since the profile is a streamline, this assumption as to the form of ϵ implies that the stream function $\psi(r,z)$ will be of the form $\psi(r,z) = R(r)\, e^{kz}$. There appears at first sight to be a contradiction here in that equation (8) defines an interface that is independent of time, whereas the co-ordinate axes have been taken as fixed in space. However, as the inertia terms in the equations of motion are negligible, the subsequent analysis is not affected to the order of approximation considered.

The restriction that ϵ be small, means that the analysis is to be restricted to the region ED of figure 4, where the flow is nearly parallel to the walls of the tube. Since by this method the nature of the flow around the nose of the bubble is not being investigated, it will obviously not be possible to find the explicit relationship between λ and $\mu U/T$. The analysis that follows will be accurate only to the first order of small quantities, i.e. to the first order in ϵ.

Assume that there exists a solution to equation (3) of the form

$$\psi(r,z) = R(r)\, e^{kz};$$

the equation then becomes

$$(d^2/dr^2 - r^{-1}\, d/dr + k^2)^2 R(r) = 0,$$

and the solution for the stream function is easily seen to be

$$\psi(r,z) = [Akr^2J_0(kr) + BkrJ_1(kr) + Ckr^2Y_0(kr) + DkrY_1(kr)]\, e^{kz}, \tag{9}$$

where A, B, C, D are arbitrary constants, J_0 and J_1 are Bessel functions, and Y_0 and Y_1 are Weber functions, in the notation and nomenclature of Watson (1944).

It will be noticed that this method of approximation has a certain similarity to the methods of water-wave theory, particularly in the respect that to obtain a more accurate solution to the problem, it would be necessary to take a solution to (3) in a more general form, that is something like

$$\psi(r,z) = \sum_k f_k(r)\, e^{kz},$$

which immediately makes the analysis exceedingly difficult.

Recalling the original equation of motion (1) it is easily shown, by using the above solution (9) for ψ in terms of r and z, that for the pressure p_2 in the viscous fluid

$$p_2 = 2\mu[Ak^2J_0(kr) + Ck^2Y_0(kr)]\, e^{kz} + E, \tag{10}$$

where E is a numerical constant.

The expressions for the boundary conditions on the wall of the tube, equation (4), are easily obtained from equation (9) by differentiation with respect to z and r, and lead to

$$AaJ_0(ka)+BJ_1(ka)+CaY_0(ka)+DY_1(ka) = 0, \tag{11}$$

and
$$(2A+Bk)\,J_0(ka)-AkaJ_1(ka)+(2C+Dk)\,Y_0(ka)-CkaY_1(ka) = 0, \tag{12}$$

respectively.

For the two stress conditions, since the flow is nearly parallel to the walls of the tube, the angle θ is approximately $\frac{1}{2}\pi$. Hence equation (5) becomes

$$\partial^2\psi/\partial r^2 - r^{-1}\,\partial\psi/\partial r - \partial^2\psi/\partial z^2 = 0 \quad \text{on} \quad r = \lambda' - \epsilon,$$

which, on substitution for the stream function, gives

$$[AkrJ_0(kr)+(A+Bk)\,J_1(kr)+CkrY_0(kr)+(C+Dk)\,Y_1(kr)]_{r=\lambda'-\epsilon} = 0.$$

Since the transverse velocities in the region ED are of the order of ϵ, so then is the stream function $\psi(r,z)$, which in turn implies that the arbitrary constants in equation (9) are also of order ϵ. Therefore, as ϵ is considered small, products of type ϵA, etc., may be neglected. Hence, by using Taylor series expansions for the Bessel and Weber function about $r = \lambda'$ and neglecting terms of the second and higher orders, the tangential condition on the profile reduces to

$$Ak\lambda' J_0(k\lambda')+(A+Bk)\,J_1(k\lambda')+Ck\lambda' Y_0(k\lambda')+(C+Dk)\,Y_1(k\lambda') = 0. \tag{13}$$

Equation (6) for the normal stress condition can be reduced by the same methods. To first-order quantities, the mean curvature κ is related to λ' and ϵ by

$$\kappa = \frac{1}{\lambda'}+\frac{\epsilon}{\lambda'^2}+\frac{d^2\epsilon}{dz^2}+O(\epsilon^2).$$

Hence substitution in (6) for ψ and p from (9) and (10) respectively, and proceeding as above, gives, to the first order in ϵ,

$$-Bk^3J_0(k\lambda')+\left(Ak^3\lambda'+\frac{Bk^2}{\lambda'}\right)J_1(k\lambda')-Dk^3Y_0(k\lambda')+\left(Ck^3\lambda'+\frac{Dk^2}{\lambda'}\right)Y_1(k\lambda')$$
$$+\frac{T\epsilon_0}{2\mu}\left(\frac{1}{\lambda'^2}+k^2\right) = \frac{1}{2\mu}\left(p_1-\frac{T}{\lambda'}-E\right)e^{-kz} = 0, \tag{14}$$

by considering the flow where the bubble is parallel to the wall of the tube.

Finally, condition (7) is similarly reduced. Division by $\sin\theta$ gives

$$-\frac{1}{r}\frac{\partial\psi}{\partial r}\cot\theta+\frac{1}{r}\frac{\partial\psi}{\partial z} = U\cot\theta.$$

Since the profile is given by $r = \lambda'-\epsilon$, it follows that the angle θ is related to ϵ by $\cot\theta = d\epsilon/dz$. Again using the order of magnitude argument, since ψ is of the order of ϵ, $r^{-1}(\partial\psi/\partial r)\cot\theta$ is of the second order of small quantities, and so is negligible compared with the other terms of the above equation. Hence, for a bubble moving with uniform velocity U down the tube,

$$[r^{-1}\,\partial\psi/\partial z]_{r=\lambda'-\epsilon} = U\,d\epsilon/\partial z,$$

to the first order. This, on similar substitution and simplification, as applied above, reduces to

$$Ak\lambda' J_0(k\lambda') + BkJ_1(k\lambda') + Ck\lambda' Y_0(k\lambda') + DkY_1(k\lambda') - U\epsilon_0 = 0, \tag{15}$$

to the first order.

Hence for a unique non-trivial solution we have, from these five equations (11) to (15), with a little rearrangement,

$$\begin{vmatrix} 1 & J_1(k\lambda a) & 0 & Y_1(k\lambda a) & 0 \\ -1/k\lambda a & J_0(k\lambda a) & J_1(k\lambda a) & Y_0(k\lambda a) & Y_1(k\lambda a) \\ \tfrac{1}{2}S\{1+(k\lambda a)^{-2}\} - k\lambda a & 0 & J_1(k\lambda a) - k\lambda a J_0(k\lambda a) & 0 & Y_1(k\lambda a) - k\lambda a Y_0(k\lambda a) \\ 0 & J_0(ka) & \lambda J_1(ka) & Y_0(ka) & \lambda Y_1(ka) \\ 0 & 2J_0(ka) - kaJ_1(ka) & k\lambda a J_0(ka) & 2Y_0(ka) - kaY_1(ka) & k\lambda a Y_0(ka) \end{vmatrix} = 0, \tag{16}$$

where λ' has been replaced by λa so that λ is the asymptotic ratio of bubble radius to inner tube radius, and $1/S = \mu U/T$.

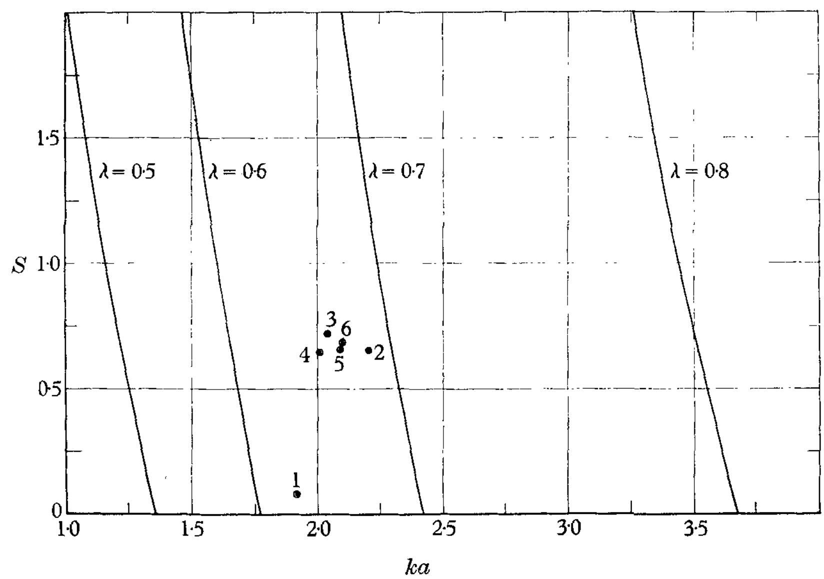

FIGURE 5. Graph plotted from equation (16). Theoretical results shown —; points marked ⊙ are taken from table 1 and given the appropriate number.

Equation (16) is effectively an implicit relation between the fundamental quantities S, λ, and k of the problem. An electronic computer, EDSAC II, was used to obtain corresponding values of S and ka (ka rather than k as then all three quantities are non-dimensional) for particular values of λ, and from these the graph shown in figure 5 was drawn. (The experimental values, obtained by the method described in § 4, are also shown for comparison.) Notice that in the theory there is no restriction on the values of λ which now lies in the range $0 \leqslant \lambda \leqslant 1$, though in practical cases λ lies between 0·6 and 1·0, from § 2.1.

As there are three parameters λ, ka and S appearing in (16), there are several ways of reproducing graphically the relationship between them. The choice made here was influenced by the requirement that the experimental points should be distinguishable from one another. An alternative representation, plotting $1-\lambda^2$ against $1/S$, i.e. $\mu U/T$, is reproduced in figure 6 and discussed in § 4.

3.4. *Linearizing approximations*

The approximations used to arrive at the solution were: (i) that $\theta \simeq \frac{1}{2}\pi$, (ii) that $\epsilon/\lambda \ll 1$, and (iii) that $k\epsilon \ll 1$. These can be grouped together in the one condition that $\epsilon \ll 1$, unless k is very large. When viscous forces are of the order of, or predominate over, surface tension forces, i.e. $\mu U/T \geqslant 1$, there is little variation in λ, and in ka. Hence, the solution may be expected to be valid until fairly near the nose of the bubble. However, for those cases when the tube is narrow, or $\mu U/T$ is small, k can become quite large, and so the analysis will probably break down. Except for these extreme cases, it is most likely that the solution is valid to within half a tube diameter of the nose of the bubble.

4. Further experiments

Since the apparatus was primarily designed for checking Taylor's (1961) hypothesis by operating at large values of $\mu U/T$ (or small S), some small modifications were necessary to obtain smaller values of this parameter (or larger values of S), so that the theory of § 3 could be tested. The main change was the insertion of a throttle valve in the mercury line at X (see figure 1) enabling the adjustment of the velocity of the bubble, U, to be made more easily. This valve also facilitated the general running of the experiment, although it made the calculation for the required initial pressure more complicated because of the pressure drop through the valve. In fact, it was usually necessary to make several preliminary runs to determine the required height of mercury and corresponding valve setting, before a bubble moving with sufficiently constant velocity had been obtained. It is a constant pressure gradient between the nose of the bubble and the end of the tube that is required.

To reach values of S of the order of 1, a higher operating temperature (30 °C) was also required. However, at this temperature there is a greater difference in the densities of CCl_4 and Golden Syrup, and the velocity of the bubble had to be greater than $\frac{1}{2}$ cm/sec for the profile to be axisymmetric. Hence it was not possible to reach values of S greater than 1 without considerable modification of the apparatus, and, for various reasons, this was not considered justifiable.

Otherwise a photograph was taken and measured as before. An exponential curve was fitted to the profile by the method described below, and thus ka found, where a was the radius of the tube in the photograph. The asymptotic ratio of the radii, λ, and the parameter S were found precisely as in § 2. The results of this for six cases are shown in table 1, and plotted graphically in figure 5. Since the photographs for small values of S were identical, only one was measured by this method, and the result shown here.

No.	k (mm^{-1})	a (mm)	ka	S	λ_{expt}	λ_{calc}
1	0·160	11·96	1·92	0·080	0·634	0·631
2	0·197	11·20	2·20	0·654	0·660	0·688
3	0·182	11·24	2·04	0·719	0·665	0·668
4	0·180	11·16	2·01	0·645	0·660	0·662
5	0·186	11·22	2·09	0·658	0·660	0·674
6	0·187	11·23	2·10	0·685	0·660	0·676

TABLE 1. Summary of experimental results of §4, for comparison with figures 5 and 6.

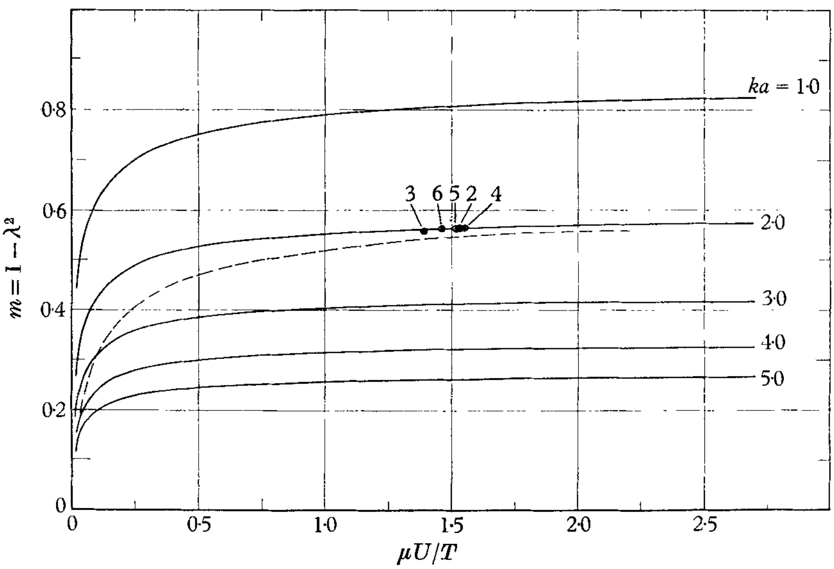

FIGURE 6. Alternative representation of equation (16), showing also the results of table 1 (as ⊙), and Taylor's experimental curve (---) for the fractional amount of fluid left in the tube as a function of $\mu U/T$.

The final column of table 1 contains the theoretical values of λ to be compared with the experimental results in the adjacent column. These were calculated by inserting the appropriate values of ka and S from the preceeding columns into equation (16), and finding the root of λ by inverse interpolation on the electronic computer.

As rather a sideline, the theoretical results of § 3 were compared with the experimental curve comparing $m = 1-\lambda^2$ with $\mu U/T$ obtained by Taylor (1961).

Rearrangement of the results obtained from the electronic computer from which figure 5 was drawn, and some further programming, made it possible to plot m against $\mu U/T$ for constant ka, as shown in figure 6. The experimental curve, taken from Taylor's work and the experimental values of table 1 are also shown. The most likely explanation of the small discrepancy between the results of table 1 and Taylor's curve, is that the slight asymmetry of the interface would give a low value for λ, and thus too large a value for m. The consistency of the error supports this view. The difference is thought to be too small to affect seriously the validity of the experimental results.

However, no practical results have been obtained from this comparison so far, except for the obvious one that ka tends to a constant value (near 2·0) as $\mu U/T$ increases. This is to be expected since λ is also tending to an asymptotic value, that is, the shape of the profile can be expected to be independent of $\mu U/T$ for large values of this parameter. This is confirmed by figure 6.

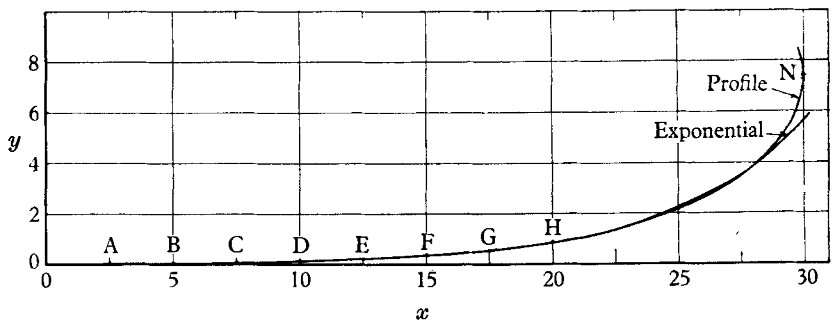

FIGURE 7. Graphical comparison of lower half of a bubble profile, and the fitted exponential, plotted from the figures given in table 2.

It is hoped, if an analytic solution is obtained for the nose of the bubble, that this will provide the necessary connexion between the three parameters λ, ka, and $\mu U/T$, thus overcoming the present indeterminate nature of this problem.

Exponential fitting

For the results in table 1 the following method was used. Eight points (A, B, ..., H) in figure 7 were chosen from the profile, the horizontal distance between H and the nose of the bubble N being the same in each photograph measured. The exponential curve $y = a\,e^{kx}+b$ was fitted to these points by the method of least squares, where a, b and k were all allowed to vary. This calculation was carried out on the electronic computer for accuracy and convenience. An example of the resulting curve, and the profile from which it was taken, is shown in figure 7. However the agreement between the two curves is so close that the values from which figure 7 was plotted are shown in table 2. It is interesting that such a large part of the profile is accurately exponential in form. In each case the value of b obtained from the computer was checked with the known value of the asymptotic radius of the bubble, and in no case did it exceed the experimental spread of the measurements from which the mean asymptotic bubble width was calculated.

x (mm)	Measured distance of profile from x-axis (mm)	Calculated distance (mm)
$-\infty$	0	$-0{\cdot}011$
0	0·015	0·010
2·5	0·018	0·021
5·0	0·040	0·041
7·5	0·077	0·072
10·0	0·124	0·121
12·5	0·203	0·201
15·0	0·320	0·328
17·5	0·531	0·530
20·0	0·851	0·853
22·5	1·342	1·369
25·0	2·130	2·194
27·5	3·460	3·511
30·0	7·409	5·615

TABLE 2. Co-ordinates giving shape of profile, and corresponding values for fitted exponential.

5. Discussion of results

(i) The primary object of this experiment was achieved, namely to find the fractional amount of fluid left sticking to the wall of a tube for large values of $\mu U/T$. The experiments described in § 2 were all at values of $\mu U/T$ greater than 10, whereas the highest value previously considered was 2·0. Hence the value of 0·60 obtained for this fraction is a most reasonable confirmation of Taylor's suggestion that it would be a little greater than 0·56. This result implies that more than half of a very viscous fluid is left in the tube.

(ii) On consideration of table 1 and figure 5, it would seem that the theory presented in § 3 is reasonable for the part of a profile where (*a*) it is of exponential form, and (*b*) the approximations in § 3 are justifiable. The close fit to an exponential curve shown by figure 7 suggests that these conditions are not as restrictive as would appear at first sight. Hence it may be assumed that equation (16) represents a solution valid under normal circumstances to within half a tube diameter of the nose of the bubble. The most important implication of this correlation between the linearized theory and experiment is that it gives confidence in the above method of selecting and using the boundary conditions. It also seems that it is justifiable to assume a well-defined surface tension for this type of problem, and for the fluids used. This is interesting as numerous workers have commented on the difficulty of using concentrated sucrose solutions, particularly as regards surface contamination, and other similar effects. As the above experiments were made in the region where viscous forces are, at the very least, of the same order of magnitude as those due to surface tension, it would be surprising to find surface effects playing an important role. This is not, however, the case in some similar problems, particularly where $\mu U/T$ is very small, as for example those discussed by Bretherton (1961).

An attempt was made to correlate this theoretical work with that of Bretherton, by calculating the value of the determinant (16) for λ very nearly equal to 1, and substituting his theoretical results in that expression. However, his results give a very large value for k, and hence the restriction that $k\epsilon$ be small, and that second-order terms may be neglected, is no longer observed. It is most probably for this reason that the attempt failed. It is in fact not particularly surprising that it did, as Bretherton's work only applies to the region $\mu U/T < 5 \times 10^{-3}$, i.e. $S > 200$.

A most noticeable feature of the experiment was the extreme stability of the profile of the bubble. Within a few tube diameters of entering the tube, the profile had attained its axisymmetric shape, and thereafter retained this bullet-shaped form throughout the rest of its passage down the tube. The majority of the apparatus had been constructed of glass or Perspex partly so that this question of the stability of the profile could be observed. The stability of the motion also seemed to apply to the initial stages. As the motion of the bubble was started by removing a cork at the end of the tube, it is not very likely that the initial conditions were identical each time. Yet as far as could be observed the initial motion was the same on every occasion. It is interesting that when Saffman & Taylor (1958) studied the relatively similar two dimensional analogue of this problem in a Hele–Shaw cell, they found that special precautions were necessary to avoid forming fingers of inviscid fluid during the initial stages of the motion. On the other hand, no particular precautions were taken in the above experiments, nor were any fingers observed.

Further work is now being carried out on a theoretical account of the nose of the bubble, but the results are not yet conclusive. An attempt is also being made to solve the complete problem numerically, with the aid of an electronic computer.

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